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# RAYLEIGH SERIES FOR DIELECTRIC WAVEGUIDES OF COMPLEX CROSS-SECTION

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***Sotsky A. B., An Ying.*** **RAYLEIGH SERIES FOR DIELECTRIC WAVEGUIDES OF COMPLEX CROSS-SECTION.**

*With the use of the Rayleigh series in cylindrical functions for the longitudinal components of the electromagnetic field and the Green's theorem, an algebraic formulation of the problem of calculating the modes of three-dimensional open dielectric waveguides of complex cross-section is obtained. The effectiveness of the approach is demonstrated by examples of calculating dielectric waveguides with elliptical cross-sections. The illustrations of the internal convergence of the method are given, and the obtained dispersion characteristics of the modes are compared with the data of other less general numerical approaches.*

**Keywords:** dielectric waveguide of complex cross-section, Rayleigh series, Green's theorem, elliptical waveguides.

## Introduction

At present, open dielectric waveguides are widely used in optical communication and information systems [1]. However, the modes of such waveguides can be rigorously calculated only in the simplest cases of planar structures and structures with circular symmetry [2–4]. This stimulates the development of numerical methods for calculating the mode characteristics of open waveguides.

Of main interest are vector methods that allow a consistently refined solution of the waveguide problem to be obtained. Currently there are three groups of such methods [5–9].

The first group includes variational methods [3; 6; 9]. Strictly speaking, they are applicable only to the calculation of the guided modes of waveguides, since the variational principle requires square-integrable mode fields in the plane orthogonal to the waveguide axis. Variational methods are subdivided into approximate analytical ones having closed formulation [3] and more accurate numerical methods [6; 9]. In numerical methods to overcome the above limitation, perfectly matched layers (PMLs) are used that completely absorb the incident radiation, simulating an open space [5; 6]. However, in the case of open three-dimensional waveguides, this approach introduces uncontrollable errors, since only flat PMLs can completely absorb the radiation. Nevertheless, the variational method known as the finite element method is widely used to estimate the characteristics of the modes of open three-dimensional waveguides [6; 9]. This method is reduced to solving by the reduction method of infinite homogeneous algebraic systems with respect to the values of the components of the electromagnetic field in the interpolation nodes, the grid of which covers the cross section of the waveguide. Because it contains such uncertainties as errors of the reduction method and the use of PMLs, the boundaries of the applicability of such obtained estimates are not always clear.

The second group includes finite difference schemes [5; 8]. They imply a replacing the differential operators in the Maxwell's system by finite differences. Boundary conditions in the schemes are formulated with the use of PMLs, modeling the open space. As above, the waveguide problem is reduced to a solving a infinite homogeneous algebraic system for values of the mode field components in interpolation nodes. As a result, finite difference schemes have qualitative the same restrictions as the finite element method.

Methods of third group are methods of integral equations and Green's function [3]. They are determined as for guided well as for leaky modes of open dielectric waveguides and do not use PMLs. So these methods allow overcome one of above restrictions. But they also use infinite algebraic systems solved by reduction schemes.

The Green's function method has been developed for microstructured optical fibers formed in a dielectric matrix by air channels of circular cross section. In this case mode field components are represented by rows in cylindrical functions and the standard Graph addition theorem for the two dimensional Green's function is used. But if the channels have more complicated cross sections this approach loses its applicability. In this case it is naturally to trade to use the mode electromagnetic field presentation by Rayleigh series in cylindrical functions. So far, such series have been used only for rectangular dielectric waveguides in the so-called collocation method [2]. In this method the boundary conditions for the electromagnetic field along the waveguide cross section are approximately satisfied by the method of least squares [2].

It is also necessary to note the method of separation of variables, applicable to waveguides of elliptic cross section [5, 10]. But in this method calculations of the Mathieu functions and infinite matrix determinant are complex computational problems that have to be solved numerically, so it cannot be called analytical.

In this work a new full vectorial mode solver that combine the Green's function method and a field presentation by Rayleigh series is studied. This approach has extended area of applicability. In particular, it allows modes of elliptic dielectric waveguides to be investigated. On the other hand, this waveguide can be investigated by other numerical methods named above. Comparison of the results of various methods for modal characteristics of elliptical dielectric waveguides opens up opportunities to assess the applicability and convergence of various computational schemes, including the new method indicated above.



### Formulation of the Green's function method

A cross-section of the waveguide under consideration is presented in Fig. 1. The waveguide is supposed to be infinite and regular in z direction. The inner and outer areas of the waveguide have relative permittivities $\varepsilon_w$ and $\varepsilon_s$, respectively. Media in these areas are supposed to be non-magnetic, i. e. their magnetic permeability is equal to the magnetic permeability of vacuum $\mu$. A mode with the time and z - dependency factor

$$\exp(i\omega t - ik_0\beta z) \tag{1}$$

propagates along the waveguide. Here $k_0 = \omega / c = 2\pi / \lambda$ is the wave number of vacuum, β is dimensionless mode propagation constant. Values of $\varepsilon_w$, $\varepsilon_s$ and β in general case can be complex.

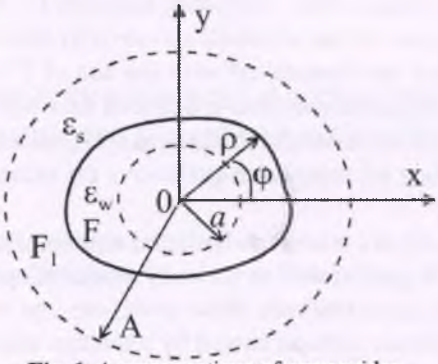


**Fig. 1.** A cross sections of waveguides under investigation

The electromagnetic field of the mode everywhere obeys the homogeneous Maxwell's equations [4]

$$\nabla \times \mathbf{H} = i\omega\varepsilon(x,y)\varepsilon_0\mathbf{E}, \tag{2}$$

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}, \tag{3}$$

where $\varepsilon(x,y)$ is a step-like function of $x$ and $y$, equal to $\varepsilon_w$ and $\varepsilon_s$ within and out the waveguide, $\varepsilon_0$ is the permittivity of vacuum.

According to (1), the vectorial equations (2), (3) are equivalent to six scalar equations

$$\frac{\partial H_z}{\partial y} + i\beta H_y = i\omega\varepsilon\varepsilon_0 E_x, \tag{4}$$

$$-i\beta H_x - \frac{\partial H_z}{\partial x} = i\omega\varepsilon\varepsilon_0 E_y, \tag{5}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega\varepsilon\varepsilon_0 E_z, \tag{6}$$

$$\frac{\partial E_z}{\partial y} + i\beta E_y = -i\omega\mu H_x, \tag{7}$$

$$i\beta E_x + \frac{\partial E_z}{\partial x} = i\omega\mu H_y, \tag{8}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu H_z, \tag{9}$$

It follows from (4), (5), (7), (8), that

$$E_x = -\frac{i}{\chi^2}\left(k_0\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right), \tag{10}$$

$$E_y = -\frac{i}{\chi^2}\left(k_0\beta\frac{\partial E_z}{\partial x} - \omega\mu\frac{\partial H_z}{\partial x}\right), \tag{11}$$

$$H_x = -\frac{i}{\chi^2}\left(k_0\beta\frac{\partial H_z}{\partial x} - \omega\varepsilon\varepsilon_0\frac{\partial E_z}{\partial y}\right), \tag{12}$$

$$H_y = -\frac{i}{\chi^2}\left(k_0\beta\frac{\partial H_z}{\partial y} + \omega\varepsilon\varepsilon_0\frac{\partial E_z}{\partial x}\right), \tag{13}$$

where $\chi^2 = k_0^2(\varepsilon - \beta^2)$. The substitution of (10), (11) into (9) and of (12), (13) into (6) yields the system of two second-order differential equations with respect to $E_z$ and $H_z$:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_0^2\kappa_s^2 E_z = f_1, \tag{14}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k_0^2\kappa_s^2 H_z = f_2, \tag{15}$$

where $\kappa_s^2 = \varepsilon_s - \beta^2$,

$$f_1 = -\frac{\chi^2}{\varepsilon\varepsilon_0}\left\{\frac{k_0\beta}{\omega}\left[\frac{\partial\chi^{-2}}{\partial x}\frac{\partial H_z}{\partial y} - \frac{\partial\chi^{-2}}{\partial y}\frac{\partial H_z}{\partial x}\right] + \frac{\partial(\varepsilon\varepsilon_0\chi^{-2})}{\partial x}\frac{\partial E_z}{\partial x} + \frac{\partial(\varepsilon\varepsilon_0\chi^{-2})}{\partial y}\frac{\partial E_z}{\partial y}\right\} - k_0^2(\varepsilon - \varepsilon_s)E_z, \tag{16}$$

$$f_2 = -\frac{k_0\beta}{\omega\mu}\left[\frac{\partial\chi^{-2}}{\partial y}\frac{\partial E_z}{\partial x} - \frac{\partial\chi^{-2}}{\partial x}\frac{\partial E_z}{\partial y}\right] - \frac{\partial\chi^{-2}}{\partial x}\frac{\partial H_z}{\partial x} - \frac{\partial\chi^{-2}}{\partial y}\frac{\partial H_z}{\partial y} - k_0^2(\varepsilon - \varepsilon_s)H_z. \tag{17}$$

So, the waveguide problem is reduced to solving the system (14), (15) with respect to the longitudinal components of the mode electromagnetic field. Next computing the total mode field can be made by formulas (10) – (13).

According to Eqs (16), (17), Eqs (14), (15) are mutually coupled on the boundary of the waveguide only. Within the waveguide they separate into two independent equations

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_0^2\kappa_w^2 E_z = 0, \tag{18}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k_0^2\kappa_w^2 H_z = 0, \tag{19}$$

where $\kappa_w = \sqrt{\varepsilon_w - \beta^2}$. Similarly, out the waveguide

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_0^2\kappa_s^2 E_z = 0, \tag{20}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + k_0^2\kappa_s^2 H_z = 0. \tag{21}$$

where $\kappa_s = \sqrt{\varepsilon_s - \beta^2}$.

In the waveguide area $r = \sqrt{x^2 + y^2} < \rho(\varphi)$ (see Fig. 1) finite solutions of Eqs (18), (19) can be presented as series in the cylindrical functions [2]

$$E_z = \sum_{\nu=-m}^{m} e_\nu^- \exp(i\nu\varphi) J_\nu(\kappa_w k_0 r), \tag{22}$$

$$H_z = \sum_{\nu=-m}^{m} h_\nu^- \exp(i\nu\varphi) J_\nu(\kappa_w k_0 r), \tag{23}$$

where $J_\nu(\kappa_w k_0 \mathrm{r})$ are the Bessel's functions of integer orders, $r$ and $\varphi$ are the polar coordinates (see Fig. 1), $e_\nu^-$, $h_\nu^-$ are unknown coefficients, $m$ is the order of the series reduction. The function $\rho(\varphi)$ describes the perimeter of the waveguide cross section (see Fig. 1).

Out the waveguide series analogous to (22), (23) are [2]:

$$E_z = \sum_{\nu=-m}^{m} e_\nu^+ \exp(i\nu\varphi) H_\nu^{(2)}(\kappa_s k_0 r), \tag{24}$$

$$H_z = \sum_{\nu=-m}^{m} h_\nu^+ \exp(i\nu\varphi) H_\nu^{(2)}(\kappa_s k_0 r), \tag{25}$$

where $e_\nu^+$, $h_\nu^+$ are unknown coefficients, $\kappa_s = \sqrt{\varepsilon_s - \beta^2}$ are the Hankel's function of the second kind and integer orders. Brunch of the radical $\kappa_s = \sqrt{\varepsilon_s - \beta^2}$ depends on the type of the mode under investigation. If the mode is guided, then $\mathrm{Im}(\kappa_s) < 0$, and if the mode is leaky, then $\mathrm{Re}(\kappa_s) > 0$ [3].

In the literature, to solve the waveguide problem using Rayleigh's series (22) – (25), the collocation method was used [2]. In it, after substituting (22) – (25) into (10) – (13), the tangential components of the electromagnetic field on the contour of the waveguide boundary are calculated within the limits from the sides of the waveguide area and its surrounding. Next, the mean-square functional of the residual of these components is constructed. From the requirement of the minimum of the functional, a homogeneous algebraic system of dimension 4 (2m + 1) x4 (2m + 1) with respect to the values of $h_\nu^\pm$, $e_\nu^\pm$ is obtained. The propagation constants $\beta$ of the waveguide modes are determined from the condition that the determinant of this system is equal to zero. Next, the mode fields are calculated using formulas (10) – (13) after expressing all amplitudes through one of them as a result of solving an inhomogeneous algebraic system with a matrix of rank 8m + 3.

The collocation method is applicable to the study of open waveguides with a simple cross-sectional shape. In more complex cases, for example, in the study of microstructured fibers, it loses its applicability. In this thesis, a more general formulation of a boundary value problem based on Green's theorem is used, free from the indicated restrictions.

To solve the system (14), (15) the two-dimensional electrodynamics Green's function [3]

$$G = \frac{i}{4} H_0^{(2)}(\kappa_s k_0 R), \tag{26}$$

where $R = |\mathbf{r}' - \mathbf{r}|$, is used. This function obeys the equation [3]

$$(\Delta' + k_0^2 \kappa_s^2) G = \delta(\mathbf{r}' - \mathbf{r}), \tag{27}$$

where $\Delta' = \partial^2 / \partial x'^2 + \partial^2 / \partial y'^2$, $\delta(\mathbf{r}' - \mathbf{r})$ is the two - dimensional Dirac's delta function.

Let's multiply Eq. (15) by $G$, Eq. (27) – by $H_z$ and integrate the results over the circular area $F_1$ of radius $A$ covering the waveguide (see Fig.1). Then subtract them term by term from each other. As a result,

$$\int_{F_1} [G\Delta' H_z(x',y') - H_z(x',y')\Delta' G] dx'dy' = \int_{F_1} f_2(x',y') G dx'dy' - \begin{cases} H_z(x,y) & in \quad F_1 \\ 0 & out \quad of \quad F_1 \end{cases}. \qquad (28)$$

Application to the left-hand part of Eq. (28) the Green's theorem [11] gives

$$A\int_0^{2\pi} \left( G\frac{\partial H_z}{\partial r'} - H_z \frac{\partial G}{\partial r'} \right)_{r'=A} d\varphi' = \int_{F_1} f\, G dx'dy' - \begin{cases} H_z(x,y) & in \quad F_1 \\ 0 & out \quad of \quad F_1 \end{cases}, \qquad (29)$$

where $x' = A\cos\varphi'$, $y' = A\sin\varphi'$, $\partial / \partial n'$ means derivative by outer normal to the area $F_1$ boundary.

Since the function (16) differs from zero in the waveguide area $F$ only, by analogy with (29) one can obtain next formula

$$\oint \left( G\frac{\partial H_z}{\partial n'} - H_z \frac{\partial G}{\partial n'} \right)_{r'=\rho(\varphi')+0} d\tau' = \int_{F_1} f\, G dx'dy' - \begin{cases} H_z(x,y) & in \quad F \\ 0 & out \quad of \quad F \end{cases}, \qquad (30)$$

where $d\tau'$ is the infinitizimal length of the waveguide area perimeter (see Fig. 1) and $\partial / \partial n'$ means derivative by outer normal to the area $F$ boundary. It should be noted that the integrand in the left hand part of Eq (30) should be taken in the limit

$$r' \to \rho(\varphi') + 0. \qquad (31)$$

For practical calculations it is convenient to write the left side of equation (2.1330) in polar coordinates $r' = \sqrt{x'^2 + y'^2}$ and $\varphi'$:

$$\oint \left( G\frac{\partial H_z}{\partial n'} - H_z \frac{\partial G}{\partial n'} \right)_{r'=\rho(\varphi')+0} d\tau' = \int_0^{2\pi} \sqrt{\left(\frac{d\rho}{d\varphi'}\right)^2 + \rho^2} \left( G\frac{\partial H_z}{\partial n'} - H_z \frac{\partial G}{\partial n'} \right)_{r'=\rho(\varphi')+0} d\varphi', \qquad (32)$$

where integrands are calculated in the limit (31) and

$$\frac{\partial}{\partial n'} = \frac{1}{\sqrt{\left(\frac{d\rho}{d\varphi'}\right)^2 + \rho^2}} \left( \rho \frac{\partial}{\partial r'} - \frac{1}{\rho}\frac{d\rho}{d\varphi'}\frac{\partial}{\partial \varphi'} \right). \qquad (33)$$

Excluding the term

$$\int_{F_1} f\, G dx'dy'$$

from Eqs (29), (30), we come to the functional equation

$$\int_0^{2\pi}\sqrt{\left(\frac{d\rho}{d\varphi'}\right)^2+\rho^2}\left(G\frac{\partial H_z}{\partial n'}-H_z\frac{\partial G}{\partial n'}\right)_{r'=\rho(\varphi')+0}d\varphi'-$$
$$-A\int_0^{2\pi}\left(G\frac{\partial H_z}{\partial r'}-H_z\frac{\partial G}{\partial r'}\right)_{r'=A}d\varphi'=\begin{cases}0 & in \quad F\\ 0 & out \quad of \quad F_1.\end{cases} \tag{34}$$

The analogous analysis of Eq. (14) results in the functional equation analogous to (34):

$$\int_0^{2\pi}\sqrt{\left(\frac{d\rho}{d\varphi'}\right)^2+\rho^2}\left(G\frac{\partial E_z}{\partial n'}-E_z\frac{\partial G}{\partial n'}\right)_{r'=\rho(\varphi')+0}d\varphi'-$$
$$-A\int_0^{2\pi}\left(G\frac{\partial E_z}{\partial r'}-E_z\frac{\partial G}{\partial r'}\right)_{r'=A}d\varphi'=\begin{cases}0 & in \quad F\\ 0 & out \quad of \quad F_1.\end{cases} \tag{35}$$

The first step towards transforming the functional equations (34), (35) into algebraic equations with respect to coefficients $h_\nu^\pm$, $e_\nu^\pm$ of the series (6) – (8) is to express through these coefficients the functions $H_z$, $\partial H_z / \partial n'$, $E_z$, $\partial E_z / \partial n'$ taken in the limit (31). Due to continuity of the electromagnetic field tangential components $E_z$ and $H_z$, on the perimeter of the waveguide, for these components series (22), (23) still be valid. To find the derivatives

$$\left(\frac{\partial H_z}{\partial n'}\right)_{r'=\rho(\varphi')+0},\quad \left(\frac{\partial E_z}{\partial n'}\right)_{r'=\rho(\varphi')+0}$$

it is sufficient to use the conditions of continuity of the tangential components of the fields **H** and **E** on the perimeter of the waveguide lied in the plane $z=0$ (see Fig. 1). Then using Eqs (10) – (13), one can obtain

$$\left(\frac{\partial H_z}{\partial n'}\right)_{r'=\rho(\varphi')+0}=\frac{\kappa_s^2}{\kappa_w^2}\left(\frac{\partial H_z}{\partial n'}\right)_{r'=\rho(\varphi')-0}+\beta\left(1-\frac{\kappa_s^2}{\kappa_w^2}\right)\left(\frac{\partial E'_z}{\partial\tau}\right)_{r'=\rho(\varphi')}, \tag{36}$$

$$\left(\frac{\partial E'_z}{\partial n'}\right)_{r'=\rho(\varphi')+0}=\frac{\kappa_s^2\varepsilon_w}{\kappa_w^2\varepsilon_s}\left(\frac{\partial E'_z}{\partial n'}\right)_{r'=\rho(\varphi')-0}-\left(1-\frac{\kappa_s^2}{\kappa_w^2}\right)\frac{\beta}{\varepsilon_s}\left(\frac{\partial H_z}{\partial\tau}\right)_{r'=\rho(\varphi')}, \tag{37}$$

where $E'_z=\sqrt{\varepsilon_0/\mu}E_z$,

$$\frac{\partial}{\partial\tau}=\frac{1}{\sqrt{\left(\frac{d\rho}{d\varphi'}\right)^2+\rho^2}}\left(\frac{d\rho}{d\varphi'}\frac{\partial}{\partial r'}+\frac{\partial}{\partial\varphi'}\right). \tag{38}$$

The second and final step consist in the use of the Graf's addition theorem consequences [11]

$$H_0^{(2)}(\kappa_s k_0 R)=\sum_{\nu=-m}^{m}\exp[\nu(\varphi-\varphi')]\begin{cases}H_\nu^{(2)}(k_0\kappa_s r')J_\nu(k_0\kappa_s r) & r<r'\\ H_\nu^{(2)}(k_0\kappa_s r)J_\nu(k_0\kappa_s r') & r>r',\end{cases} \tag{39}$$

where $r=|\mathbf{r}|$, $r'=|\mathbf{r}'|$. According to (33), (34), (36), (38), (39), within a circular area of radius $a<\rho_{\min}$, where $\rho_{\min}$ is the function $\rho(\varphi)$ minimum (see Fig.1),

$$\sum_{\nu=-m}^{m}\exp(i\nu\varphi)J_\nu(k_0\kappa_w r)\sum_{\mu=-m}^{m}(HHI_{\nu\mu}h_\mu^- + HEI_{\nu\mu}e_\mu^-)=0, \tag{40}$$

$$\begin{aligned} HHI_{\nu\mu} &= \int_0^{2\pi} d\varphi'\exp[i(\mu-\nu)\varphi']\times \\ &\times\{k_0\kappa_s\rho[(\kappa_s/(2\kappa_w))][(J_{\mu-1}(k_0\kappa_w\rho)-J_{\mu+1}(k_0\kappa_w\rho))H_\nu^{(2)}(k_0\kappa_s\rho)- \\ &-J_\mu(k_0\kappa_w\rho)(H_{\nu-1}^{(2)}(k_0\kappa_s\rho)-H_{\nu+1}^{(2)}(k_0\kappa_s\rho))]- \\ &-i\frac{1}{\rho}\frac{d\rho}{d\varphi'}(\nu+\mu\frac{\kappa_s^2}{\kappa_w^2})H_\nu^{(2)}(k_0\kappa_s\rho)J_\mu(k_0\kappa_w\rho)\}, \end{aligned} \tag{41}$$

$$\begin{aligned} HEI_{\nu\mu} &= \beta(1-\frac{\kappa_s^2}{\kappa_w^2})\int_0^{2\pi} d\varphi'\exp[i(\mu-\nu)\varphi']\times \\ &\times\{i\mu J_\mu(k_0\kappa_w\rho)+\frac{k_0\kappa_w}{2}\frac{d\rho}{d\varphi'}[(J_{\mu-1}(k_0\kappa_w\rho)-J_{\mu+1}(k_0\kappa_w\rho)]\}H_\nu^{(2)}(k_0\kappa_s\rho). \end{aligned} \tag{42}$$

Within the open area $r > A$ (see Fig.1)

$$\sum_{\nu=-m}^{m}\exp(i\nu\varphi)H_\nu^{(2)}(k_0\kappa_s r)\sum_{\mu=-m}^{m}(h_\nu^+ - HHO_{\nu\mu}h_\mu^- - HEO_{\nu\mu}e_\mu^-)=0, \tag{43}$$

$$\begin{aligned} HHO_{\nu\mu} &= \int_0^{2\pi} d\varphi'\exp[i(\mu-\nu)\varphi']\times \\ &\times\{k_0\kappa_s\rho[(\kappa_s/(2\kappa_w))][(J_{\mu-1}(k_0\kappa_w\rho)-J_{\mu+1}(k_0\kappa_w\rho))J_\nu(k_0\kappa_s\rho)- \\ &-J_\mu(k_0\kappa_w\rho)(J_{\nu-1}(k_0\kappa_s\rho)-J_{\nu+1}(k_0\kappa_s\rho))]- \\ &-i\frac{1}{\rho}\frac{d\rho}{d\varphi'}(\nu+\mu\frac{\kappa_s^2}{\kappa_w^2})J_\nu(k_0\kappa_s\rho)J_\mu(k_0\kappa_w\rho)\}, \end{aligned} \tag{44}$$

$$\begin{aligned} HEO_{\nu\mu} &= \beta(1-\frac{\kappa_s^2}{\kappa_w^2})\int_0^{2\pi} d\varphi'\exp[i(\mu-\nu)\varphi']\times \\ &\times\{i\mu J_\mu(k_0\kappa_w\rho)+\frac{k_0\kappa_w}{2}\frac{d\rho}{d\varphi'}[(J_{\mu-1}(k_0\kappa_w\rho)-J_{\mu+1}(k_0\kappa_w\rho)]\}J_\nu(k_0\kappa_s\rho). \end{aligned} \tag{45}$$

Obtaining Eqs (41) – (45) the relation

$$\frac{dZ_\nu}{dz}=0.5(Z_{\nu-1}-Z_{\nu+1}).$$

where $Z_\nu(z)$ is any cylindrical function [11], was used.

Since functions $\exp(i\nu\varphi)J_\nu(k_0\kappa_w r)$ in (40) and $\exp(i\nu\varphi)H_\nu^{(2)}(k_0\kappa_s r)$ in (43) are linearly independent, (40) and (43) result in algebraic equations

$$\sum_{\mu=-m}^{m}(HHI_{\nu\mu}h_\mu^- + HEI_{\nu\mu}e_\mu^-)=0 \quad (\nu=\overline{-m,m}), \tag{46}$$

$$h_\nu^+ = \sum_{\mu=-m}^{m} (HHO_{\nu\mu} h_\mu^- + HEO_{\nu\mu} e_\mu^-) \quad (\nu = \overline{-m,m}). \qquad (47)$$

An analogous analysis of functional equations (35) yields

$$\sum_{\mu=-m}^{m} (EHI_{\nu\mu} h_\mu^- + EEI_{\nu\mu} e_\mu^-) = 0 \quad (\nu = \overline{-m,m}), \qquad (48)$$

$$e_\nu^+ = \sum_{\mu=-m}^{m} (EHO_{\nu\mu} h_\mu^- + EEO_{\nu\mu} e_\mu^-) \quad (\nu = \overline{-m,m}), \qquad (49)$$

$$EHI_{\nu\mu} = -\frac{\beta}{\varepsilon_s}(1-\frac{\kappa_s^2}{\kappa_w^2}) \int_0^{2\pi} d\varphi' \exp[i(\mu-\nu)\varphi'] \times$$
$$\times \{ i\mu J_\mu(k_0\kappa_w\rho) + \frac{k_0\kappa_w}{2}\frac{d\rho}{d\varphi'}[(J_{\mu-1}(k_0\kappa_w\rho) - J_{\mu+1}(k_0\kappa_w\rho)]\} H_\nu^{(2)}(k_0\kappa_s\rho), \qquad (50)$$

$$EEI_{\nu\mu} = \int_0^{2\pi} d\varphi' \exp[i(\mu-\nu)\varphi'] \times$$
$$\times \{ k_0\kappa_s\rho[(\kappa_s\varepsilon_w / (2\kappa_w\varepsilon_s))[(J_{\mu-1}(k_0\kappa_w\rho) - J_{\mu+1}(k_0\kappa_w\rho)) H_\nu^{(2)}(k_0\kappa_s\rho) -$$
$$- J_\mu(k_0\kappa_w\rho)(H_{\nu-1}^{(2)}(k_0\kappa_s\rho) - H_{\nu+1}^{(2)}(k_0\kappa_s\rho))] -$$
$$- i\frac{1}{\rho}\frac{d\rho}{d\varphi'}(\nu + \mu\frac{\kappa_s^2\varepsilon_w}{\kappa_w^2\varepsilon_s}) H_\nu^{(2)}(k_0\kappa_s\rho) J_\mu(k_0\kappa_w\rho)\}, \qquad (51)$$

$$EHO_{\nu\mu} = -\frac{\beta}{\varepsilon_s}(1-\frac{\kappa_s^2}{\kappa_w^2}) \int_0^{2\pi} d\varphi' \exp[i(\mu-\nu)\varphi'] \times$$
$$\times \{ i\mu J_\mu(k_0\kappa_w\rho) + \frac{k_0\kappa_w}{2}\frac{d\rho}{d\varphi'}[(J_{\mu-1}(k_0\kappa_w\rho) - J_{\mu+1}(k_0\kappa_w\rho)]\} J_\nu(k_0\kappa_s\rho), \qquad (52)$$

$$EEO_{\nu\mu} = \int_0^{2\pi} d\varphi' \exp[i(\mu-\nu)\varphi'] \times$$
$$\times \{ k_0\kappa_s\rho[(\varepsilon_w\kappa_s / (2\kappa_w\varepsilon_s))[(J_{\mu-1}(k_0\kappa_w\rho) - J_{\mu+1}(k_0\kappa_w\rho)) J_\nu(k_0\kappa_s\rho) -$$
$$- J_\mu(k_0\kappa_w\rho)(J_{\nu-1}(k_0\kappa_s\rho) - J_{\nu+1}(k_0\kappa_s\rho))] -$$
$$- i\frac{1}{\rho}\frac{d\rho}{d\varphi'}(\nu + \mu\frac{\kappa_s^2\varepsilon_w}{\kappa_w^2\varepsilon_s}) J_\nu(k_0\kappa_s\rho) J_\mu(k_0\kappa_w\rho)\}. \qquad (53)$$

Thus, the diffraction problem on the investigation of dielectric waveguides of complex cross-section by the Green's function method is reduced to solving the homogeneous algebraic system (46), (48) with respect to coefficients of Rayleigh expansions (22), (23) for longitudinal components of the mode electromagnetic field within the waveguide area. Possible values of the mode propagation constants $\beta$ are zeros of the system (46), (48) determinant. Finding these zeros play a key role and can be performed by the contour integration method [3]. Next computing the outer mode field consists in the use of expansions (24), (25) with direct computing of amplitudes of the cylindrical functions by formulas (47), (49). The shape of the cross-section of the concrete waveguide is taken into account by the function $\rho(\varphi')$ entering the integral formulas for matrix elements (41), (42), (44), (45), (50) – (53).

To check the correctness of the developed computational scheme, let us consider the passage to the limit of an waveguide investigated into a round waveguide, for which there is an analytical solution to the waveguide problem [4]. In this case $\rho(\varphi') = a = const$. As a result, the integrals in Eqs (41), (42), (44), (45), (50) – (53) can be calculated analytically. The result is:

$$HHI_{\nu\mu} = \pi\delta_{\nu\mu}k_0\kappa_s a\{\frac{\kappa_s}{\kappa_w}[J_{\mu-1}(k_0\kappa_w a) - J_{\mu+1}(k_0\kappa_w a)]\mathrm{H}_\mu^{(2)}(k_0\kappa_s a) -$$
$$-J_\mu(k_0\kappa_w a)[\mathrm{H}_{\mu-1}^{(2)}(k_0\kappa_s a) - \mathrm{H}_{\mu+1}^{(2)}(k_0\kappa_s a)]\},$$

$$HEI_{\nu\mu} = 2\pi\delta_{\nu\mu}\beta(1-\frac{\kappa_s^2}{\kappa_w^2})i\mu J_\mu(k_0\kappa_w a)\,\mathrm{H}_\mu^{(2)}(k_0\kappa_s a)$$

$$EHI_{\nu\mu} = -2\pi\delta_{\nu\mu}\frac{\beta}{\varepsilon_s}(1-\frac{\kappa_s^2}{\kappa_w^2})i\mu J_\mu(k_0\kappa_w a)\,\mathrm{H}_\mu^{(2)}(k_0\kappa_s a)$$

$$EEI_{\nu\mu} = \pi\delta_{\nu\mu}k_0\kappa_s a\{\frac{\kappa_s\varepsilon_w}{\kappa_w\varepsilon_s}[J_{\mu-1}(k_0\kappa_w a) - J_{\mu+1}(k_0\kappa_w a)]\mathrm{H}_\mu^{(2)}(k_0\kappa_s a) -$$
$$-J_\mu(k_0\kappa_w a)[\mathrm{H}_{\mu-1}^{(2)}(k_0\kappa_s a) - \mathrm{H}_{\mu+1}^{(2)}(k_0\kappa_s a)]\},$$

where $\delta_{\nu\mu}$ is the Kroneker's delta. In this case determinant of the system (46), (48) of dimension 2x2 is

$$\det = \frac{(\kappa_s k_0 a)^2}{4}\{\frac{\kappa_s}{\kappa_w}[J_{\mu-1}(\kappa_w k_0 a) - J_{\mu+1}(\kappa_w k_0 a)]H_\mu^{(2)}(\kappa_s k_0 a) -$$
$$-J_\mu(\kappa_w k_0 a)[H_{\mu-1}^{(2)}(\kappa_s k_0 a) - H_{\mu+1}^{(2)}(\kappa_s k_0 a)]\}\times$$
$$(\frac{\kappa_s}{\kappa_w}\frac{\varepsilon}{\varepsilon_s}[J_{\mu-1}(\kappa_w k_0 a) - J_{\mu+1}(\kappa_w k_0 a)]H_\mu^{(2)}(\kappa_s k_0 a) - \quad (54)$$
$$-J_\mu(\kappa_w k_0 a)[H_{\mu-1}^{(2)}(\kappa_s k_0 a) - H_{\mu+1}^{(2)}(\kappa_s k_0 a)]\} -$$
$$-\mu^2(\frac{\kappa_s^2}{\kappa_w^2}-1)^2\frac{\beta^2}{\varepsilon_s}[J_\mu(\kappa_w k_0 a)H_\mu^{(2)}(\kappa_s k_0 a)]^2 = 0.$$

Using the new variable $\gamma = i\kappa_s$ and taking into account identities [11]

$$H_\mu^{(l)}[-(-1)^l i\gamma k_0 a] = (-1)^l\frac{2i}{\pi}\exp[(-1)^l i\mu\frac{\pi}{2}]K_\mu(\gamma k_0 a),$$

where $l = 1, 2$, one can reduce (54) to the form

$$[\frac{\varepsilon_w}{\varepsilon_s}\frac{\gamma^2 k_0 a}{2\kappa_w}\frac{J_{\mu-1}(\kappa_w k_0 a) - J_{\mu+1}(\kappa_w k_0 a)}{J_\mu(\kappa_w k_0 a)} + i\gamma k_0 a\frac{H_{\mu-1}^{(1)}(i\gamma k_0 a) - H_{\mu+1}^{(1)}(i\gamma k_0 a)}{2H_\mu^{(1)}(i\gamma k_0 a)}]\times$$
$$\times[\frac{\gamma^2 k_0 a}{2\kappa_w}\frac{J_{\mu-1}(\kappa_w k_0 a) - J_{\mu+1}(\kappa_w k_0 a)}{J_\mu(\kappa_w k_0 a)} + i\gamma k_0 a\frac{H_{\mu-1}^{(1)}(i\gamma k_0 a) - H_{\mu+1}^{(1)}(i\gamma k_0 a)}{2H_\mu^{(1)}(i\gamma k_0 a)}] =$$
$$= \frac{\mu^2\varepsilon_s\beta^2}{\kappa_w^4}(1-\frac{\varepsilon_w}{\varepsilon_s})^2,$$

that is in agreement with the classical dispersion equation for the round dielectric waveguide [4].

The limit transition obtained confirms validation of the approach developed. In the next section it to be applied to the study of modes of open dielectric waveguides of elliptical cross section.

**Computation of open dielectric waveguides of elliptical cross section**

To test the method developed in the previous section it is naturally to compare its results with ones of less general computational schemes for waveguides for which these schemes are applicable. Such waveguides include dielectric waveguides of elliptical cross section, which are of significant practical interest.

A cross section of an open dielectric elliptical waveguide is shown in inset to Figs 2 and 3. Here $a$ and $b$ are semi-major and semi-minor axes of the ellipse. In this case

$$\rho(\varphi)=\frac{ab}{\sqrt{b^2\cos^2\varphi+a^2\sin^2\varphi}},$$

$$\frac{d\rho}{d\varphi}=\frac{ab\sin(2\varphi)(b^2-a^2)}{2\left(\sqrt{b^2\cos^2\varphi+a^2\sin^2\varphi}\right)^3}.$$

As a result, matrix elements (41), (42), (44), (45), (50) – (53) have to be computed numerically only. Corresponding calculations were performed with a Fortran computer program. Integrals were computed by the Simpson's scheme. In computations of cylindrical functions the row

$$J_\nu(z)=\frac{z^{2\nu}}{2^\nu 0!\nu!}-\frac{z^{2\nu+2}}{2^{\nu+2}1!(\nu+1)!}+\frac{z^{2\nu+4}}{2^{\nu+4}2!(\nu+2)!}-\dots$$

if $|z|\le 10$, or the formula

$$J_\nu(z)=0.5[H_\nu^{(1)}(z)+H_\nu^{(2)}(z)]$$

if $|z|>10$ [11] were used. Henkel's functions were taken in the Poisson's presentation [11]

$$H_\nu^{(1)}(z)=\sqrt{\frac{2}{\pi}}\frac{(2z)^\nu\exp[i(z-\pi\nu/2-\pi/4)]}{(2\nu-1)!!\sqrt{\pi}}\int_0^\infty\frac{\exp(-zt)[t(1+it/2)]^\nu}{\sqrt{t(1+it/2)}}dt,$$

$$H_\nu^{(2)}(z)=\sqrt{\frac{2}{\pi}}\frac{(2z)^\nu\exp[-i(z-\pi\nu/2-\pi/4)]}{(2\nu-1)!!\sqrt{\pi}}\int_0^\infty\frac{\exp(-zt)[t(1+it/2)]^\nu}{\sqrt{t(1-it/2)}}dt.$$

Fig.2 illustrates the dispersion of the fundamental modes with maximal propagation constants of elliptic waveguides of different formats $\phi=b/a$. Here $B=(\beta^2-\varepsilon_s)/(\varepsilon_w-\varepsilon_s)$ is the normalized mode propagation constant.

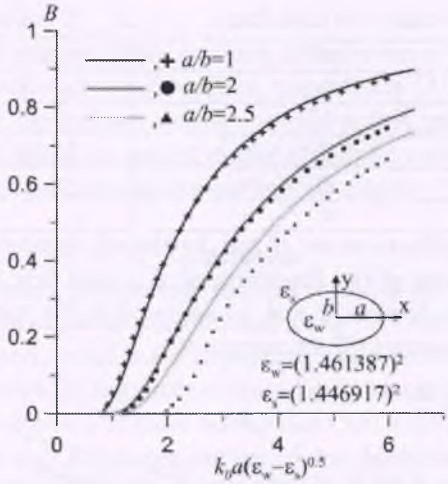


**Fig. 2.** Dispersion curves of the fundamental mode of elliptic waveguides of different formats. Solid curves – developed method, discrete points – literature data [10]

Discrete points in Fig. 2 – results of the work [10] obtained by the method of separation of variables in elliptic coordinates. This separation of variables allows electromagnetic field in the Mathieu functions to be expressed. However in this case the dispersion equation for $\beta$ has very complex form of the infinite matrix determinant. Numerically finding zeros of this determinant is not an easy task, it can be accompanied by computational errors. Apparently, it is precisely these errors that explain the discrepancy between the solid curves and discrete points in Fig. 2. In particular, for $a/b = 1$ (round waveguide) the Green's function method gives an exact analytical solution to the waveguide problem. Therefore, the systematic deviation of discrete points (crosses) from the solid curve in the upper part of the dispersion dependence for waveguide format 1 in Fig. 2 indicates errors in the computational scheme of work [10]. According to Fig. 2, these errors tend to increase with increasing the waveguide format.

High accuracy of the developed Green's function method is obvious from Table, which illustrates its internal convergence.

The internal convergence of the Green's function computational scheme in relation to the waveguide described in Fig.2 at $k_0a\sqrt{\varepsilon_w-\varepsilon_s}=4$.

| $\phi = a/b$ | m | B |
|---|---|---|
| 1 | 1 | 0.771859 |
| | 3 | 0.771859 |
| | 5 | 0.771859 |
| | 7 | 0.771859 |
| | 9 | 0.771859 |
| 2 | 1 | 0.563500 |
| | 3 | 0.590069 |
| | 5 | 0.590112 |
| | 7 | 0.590111 |
| | 9 | 0.590111 |

*Table ending*

| $\phi = a/b$ | m | B |
|---|---|---|
| 2.5 | 1 | 0.464619 |
| | 3 | 0.506356 |
| | 5 | 0.503078 |
| | 7 | 0.503074 |
| | 9 | 0.503074 |

Another test of the effectiveness of the developed approach is shown in Fig. 3, where the dispersion curves of the fundamental $E_{x11}$ and first higher $E_{x21}$ modes of elliptical waveguides with formats 2 and 3 computed in the our approach and by the numerical finite element method [9] are compared.

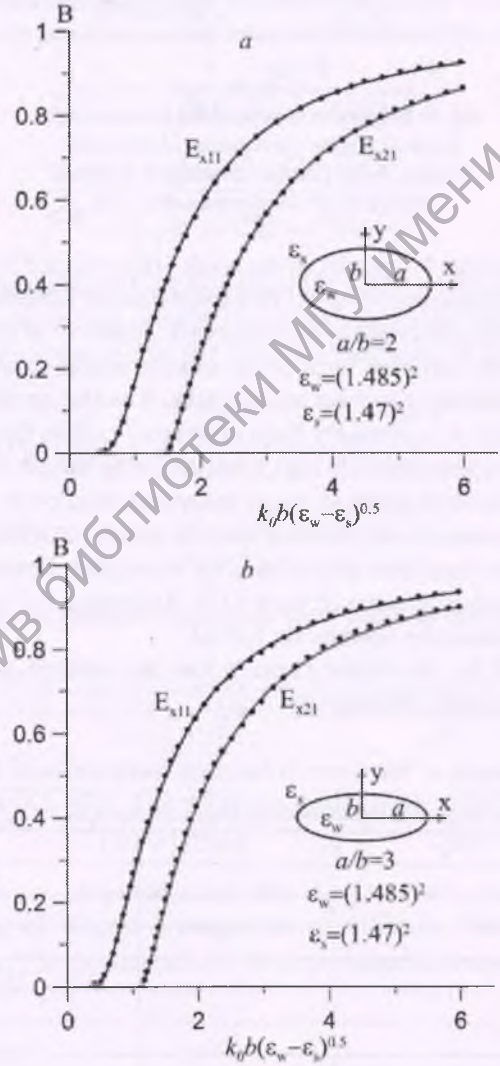


Fig. 3. Dispersion curves of the fundamental $E_x$ and the first higher $E_{x21}$ modes of elliptic waveguides of formats $\phi = 2$ (*a*) and $\phi = 3$ (*b*). Solid curves – developed method, discrete points – the finite element method [9]

The mode designation $E_{xij}$ in Fig. 3 indicates the main component of its electric field and the number of maxima in the intensity distribution of the mode over the waveguide cross section in *x* (*i*) and *y* (*j*) directions, respectively [3].

The results of both approaches almost coincide within graphic errors. As the finite element method is well suited for guided modes of elliptic waveguides [9], this comparison again confirms high precision of the method developed.

## Conclusion

As a result of the work performing, modern methods for calculating three-dimensional open dielectric waveguides were studied using the Green's theorem and Rayligh series has been developed, suitable for calculating open dielectric waveguides of complex cross-section. Details of the method have been concretized for dielectric waveguides of elliptic cross section. For these waveguides fast inner convergence of the method has been confirmed and dispersion curves of the fundamental and first high order modes have been computed and compared with known literature data. The comparison confirms high efficiency of the approach developed, which even turned out to be more accurate than the classic method of separation of variables.

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**Сотский А. Б., An Ying. РЯДЫ РЭЛЕЯ ДЛЯ ДИЭЛЕКТРИЧЕСКИХ ВОЛНОВОДОВ СЛОЖНОГО СЕЧЕНИЯ.**

*С использованием рядов Рэлея по цилиндрическим функциям для продольных компонент электромагнитного поля и теоремы Грина получена алгебраическая формулировка задачи о расчете мод трехмерных открытых диэлектрических волноводов сложного сечения. Эффективность подхода продемонстрирована на примерах расчета диэлектрических волноводов эллиптического поперечного сечения. Даны иллюстрации внутренней сходимости метода и выполнено сравнение полученных дисперсионных характеристик мод с данными других менее общих численных методов.*

**Ключевые слова:** диэлектрические волноводы сложного сечения, ряды Рэлея, теорема Грина, эллиптические волноводы.