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AXIOMATIC APPROACH TO THE TEACHING OF SOLID GEOMETRY IN GRADE IX*

Our discussion is largely concerned with the concept of reflection at a plane. (The great value of symmetry considerations in teaching is pointed out in [5] for example.) Two fundamental problems arose when we introduced this concept into the high-school course of solid geometry: The first was to decide at which point in the course to introduce it, and the second was the development of an acceptable and effective teaching methodology. The two problems are of course related and must be considered together. The usual method adopted in solid geometry textbooks is to base the discussion of reflection at a plane on the perpendicularity of straight line and planes, i.e. the approach is non-axiomatic. Although there are definite advantages in this approach, it does suffer from a very important disadvantage: the transformation is introduced quite late, so that it cannot be used throughout the theoretical course or in the solution of exercises. Axiomatic definition of reflection at a plane**, on the other hand, ensures that the transformation is introduced quite early in the solid geometry course.

Before the solid geometry course proper was started we systematized and extended the students' ideas on the deductive structure of geometry, and formulated and generalized the axioms of plane geometry.

The 9th grade solid geometry course was given on a relatively broad axiomatic base including the axioms of joining, order, parallelism, and reflection at a plane.

Axioms of reflection at a plane and their role in the construction of a solid geometry course will be discussed below in greater detail. Axioms involving reflection at a plane were approached through physical observations of reflection from a plane mirror. These experiments are simple and can be carried out not only in class but also at home. They are very helpful in providing a descriptive and not merely formal interpretation of the axioms of reflection at a plane. Although a plane mirror reflects only a half-space, it is readily verified by visual inspection that there is a point-to-point correspondence between object and image spaces, and that points apparently behind the mirror can be reflected and produce images in front of the mirror. Reflection

* Translated from *Matematika v Skole* 1969, No. 4, 60–63: 'Opyt aksiomatičeskogo izloženija kursa stereometrii IX klasse'.

** The axiomatic definition of the concept of reflection at a plane is based on axioms of reflection at a plane given in [4].

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at a plane is very effectively illustrated with the aid of a thin plane mirror coated on both sides. In the course of the plane-mirror experiments the students are guided towards an independent formulation of each reflection axiom.

Thus, the first axiom is arrived at by carrying out the following exercises.

Exercise 1. When you look into a mirror you see an image of yourself. Where is this image located: on the mirror or behind it? Verify your guess by experiment.

Hint. Touch the face of the mirror with, say, the point of a pencil, and look at the mutual disposition of the pencil's image and the plane of the mirror.

The student can readily verify that points on the surface of the mirror give rise to images that are also on the surface, whereas points well in front of the mirror correspond to images behind the mirror.

Exercise 2. Make a mark on a piece of paper and indicate it by the letter A. Reflect this point in a mirror and note the position of its image A'. Find the position of the image of A' when this in turn is reflected.

Hint. It is useful to employ a mirror coated on both sides for this experiment. The plane of the mirror should coincide with the plane of symmetry of the eyes of the observer.

This exercise demonstrates the one-to-one correspondence between object and image points: if a point A transforms into a point A' on reflection, then A' transforms into A after the same operation.

This is followed by the explicit formulation of the axiom:

Axiom 1. Each plane in space defines a reflection transformation which transforms points in one half-space into points in the other half-space and vice versa; points on the reflection plane itself remain fixed.

We shall represent the above transformation symbolically as follows:

$$A' \equiv \alpha(A),$$

which should be read as: "Reflection of a point A in plane α results in A'" or "The point A' is the image of A under α".

The other axioms of reflection at a plane can be introduced in a similar way.

Axiom 2. Reflection of a straight line at a plane results in a straight line, and the order of points remains unchanged in this transformation.

If, as a result of reflection in plane α, a straight line a is transformed into a straight line a', we write

$$a' \equiv \alpha(a).$$

Consequence 1. Reflection of a segment at a plane results in a segment, and reflection of a ray results in a ray.

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The following definitions have to be introduced:

Definition 1. The plane defining the reflection is called the reflection plane, and the two figures that correspond to each other on reflection are said to be reflection images.

In addition to the phrase "reflection at a plane" we use "mirror reflection", "symmetry with respect to a plane", and so on.

Definition 2. If α maps A into A', then the inverse α^{-1} of α maps A' into A (for any A, A').*

Definition 3. Transformations which are their own inverses are called involutions.

Theorem 1. The reflection image of a plane at a plane is a plane.

Proof. Let A, B, and C be three non-collinear points on a plane β, and let $A' \equiv \alpha(A)$, $B' \equiv \alpha(B)$, $C' \equiv \alpha(C)$ (Figure 1). Axioms 1 and 2 show that A', B', and C' are non-collinear. Let β' be the plane passing through these points. We have to prove that $\beta' \equiv \alpha(\beta)$, i.e. that the reflection of β in α is β'.

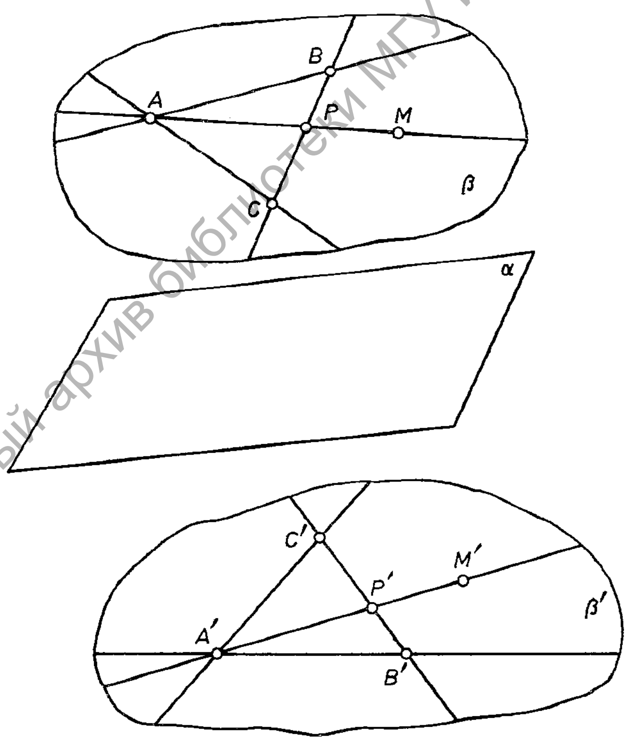

Fig. 1.

* The Russian text has been corrected.

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(1) We shall show that at α all points of β are reflected into points of β'.

If points on β are also points on the straight lines AB, BC, and CA then by Axiom 2 they transform into points lying on the straight lines $A'B'$, $B'C'$, and $C'A'$ and, consequently, into points on β'.

Suppose now that M is a point on β that is not a point on any of these straight lines. We shall suppose that the straight line AM cuts the line BC at P. Hence the point $P' \equiv \alpha(P) \in B'C' \subset \beta'$, and the straight line $A'P' \subset \beta'$, but $\alpha(M) \equiv M' \in A'P'$. Therefore $M' \in \beta'$. Thus at α all points of the plane β are reflected into points of the plane β'.

(2) It only remains to prove that each point on the plane β' is the image of a point on the plane β. To prove this we must consider the transformation in the inverse direction.

The concept of invariance is essential in the sequel.

Definition 4. Any particular relation between geometric figures that is unaffected by reflection from a plane is said to be invariant under this transformation.

The following axiom states the invariance of symmetry relations between geometric figures under all reflections in a plane.

Axiom 3. If the reflection at the plane β transforms a plane α into a plane α', then a pair of figures which are mutually symmetric with respect to α will transform into a pair of figures which are mutually symmetric with respect to α'.

The following axioms are special axioms of Euclidean geometry:

Axiom 4. Two different points in space determine one and only one reflection plane for these points.

Axiom 5. Two different rays with a common origin determine one and only one plane of reflection for these rays.

Consequence 2. Since reflection in a plane constitutes a one-to-one correspondence, and a plane reflected by a plane becomes a plane, it follows that the following relations of straight lines are invariant under all reflections in a plane: intersecting, parallelism, and skewness.

Consequence 3. If $M \in \alpha$, then $\alpha(M) \equiv M$. Conversely, if $\alpha(M) \equiv M$ then $M \in \alpha$.

Consequence 4. The point of intersection of reflection image straight lines lies on the reflection plane.

Proof. Let $\alpha(a) \equiv a'$ and $a \times a' \equiv A$. We then have $\alpha(A) \equiv \alpha(a \times a') \equiv a' \times a = A$, and hence $\alpha(A) \equiv A$, so that by Consequence 3 we have $A \in \alpha$.

Definition 5. When a figure and its image in a plane coincide, the figure is said to be invariant under this transformation. Among the invariant figures there are the fixed figures, i.e. figures each point of which is invariant on transformation.

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The following property is particularly important for the introduction of the concept of a perpendicular to a plane.

Consequence 5. A straight line passing through two different points that are reflection images of each other is an invariant line.

Proof. Let $\alpha(A) \equiv A'$. Then $\alpha(AA') \equiv A'A$, but $A'A$ and AA' are identical straight lines, so that $AA' \equiv A'A$, i.e. AA' is a straight line invariant on reflection in the given plane.

Definition 6. A straight line that is invariant under reflection at a plane, but does not belong to it, is called a perpendicular to that plane. The plane is then said to be perpendicular to the straight line.

This is readily illustrated by placing the point of a pencil in contact with the surface of the mirror and adjusting its position until the pencil and its image lie on the same straight line.

Theorem 2. A plane passing through two different points that are reflection images of each other is an invariant plane.

Proof. Let $\alpha(A) \equiv A'$, $A \in \beta$ and $A' \in \beta$ (Figure 2). We shall prove that $\alpha(\beta) \equiv \beta$. Let us consider the straight line $a \equiv AA'$. Suppose that $AA' \times \alpha \equiv B$. The planes α and β have a common point B and intersect along the straight line b. We have $\alpha(a) \equiv \alpha' \equiv a$ (Consequence 5) and $\alpha(b) \equiv b' \equiv b$ (Consequence

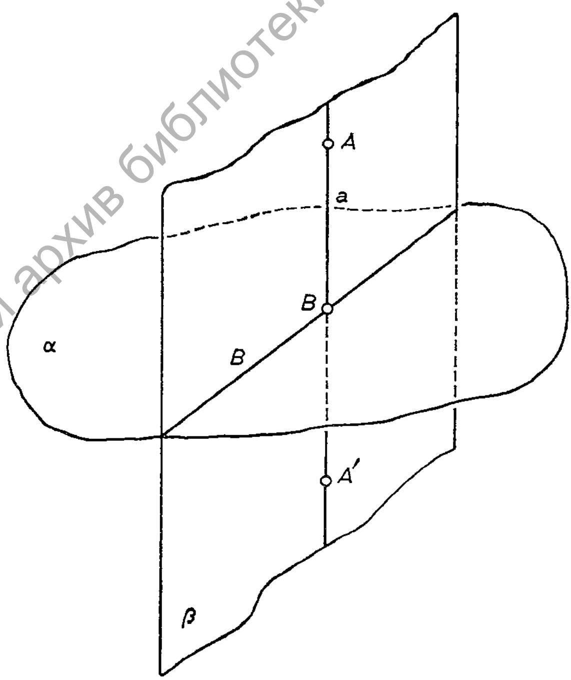

Fig. 2.

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3). Suppose that $\alpha(\beta) \equiv \beta'$; then since $a' \subset \beta'$ and $b' \subset \beta'$ whereas $b' \equiv b$ and $a' \equiv a$, it follows that $b \subset \beta'$ and $a \subset \beta'$. The planes β and β' have in common two intersecting straight lines. Therefore $\beta \equiv \beta'$.

It follows that $\alpha(\beta) \equiv \beta$.

Definition 7. A plane that is invariant under reflection at a given plane, and is different from it, is said to be perpendicular to the given plane.

Reflection at a plane is used to define the important concept of congruence of arbitrary figures.

Definition 8. Two figures are said to be congruent if one of them transforms into the other by a number of successive reflections at planes. Two figures are said to be properly congruent if one of them is transformed into the other as a result of the successive application of an even number of reflections at planes. Two figures are said to be improperly congruent if one of them is transformed into the other by a successive odd number of reflections at planes.

The students are told that the properly congruent figures are the congruent and equally oriented figures, whereas the improperly congruent figures are the congruent and oppositely oriented ones.

Definition 9. A transformation that is obtained by successive reflections at planes is called an isometry.

The class is then told that by isometry each figure can be made to "occupy" any position in space. Isometry can be used, for example, to transform any given segment into a segment on a given straight line in space. Figure 3 shows a possible method of transforming a segment AB into a segment $A'B'$ lying on a straight line l and extending from a given point A' in a given direction. This is achieved first by transforming A into A' with the aid of the reflection plane α of these two points (*Axiom* 4), and transforming the resulting segment $A'B$ by a new reflection at the reflection plane α' of l and $A'B$, into the required segment on the line l (Axiom 5). The segment AB could be transformed into the segment on l extending in a given direction from the point A' on it in the following way. First we transform B into A', and then reflect the resulting segment $A'A_1$ at the reflection plane of l and $A'A_1$ into the required segment on l.

It is possible to select an infinite set of other reflections at planes which would lead to the same result. The problem then arises as to what is the mutual disposition of the segments obtained in this way on the straight line l. Experiment shows that these segments coincide. We thus arrive at the last axiom, which together with Axioms 1–5 completely covers the concept of reflection at a plane.

Axiom 6. Whatever the isometry that is used to transform a given segment into a segment lying on a given straight line, and extending from a given point on this line in a given direction, we always obtain the same segment.

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Thus, Axiom 6 states that the segment extending on a given line from a given point in a given direction, and congruent with a given segment, is uniquely determined.*

This is then used to derive the following consequences.

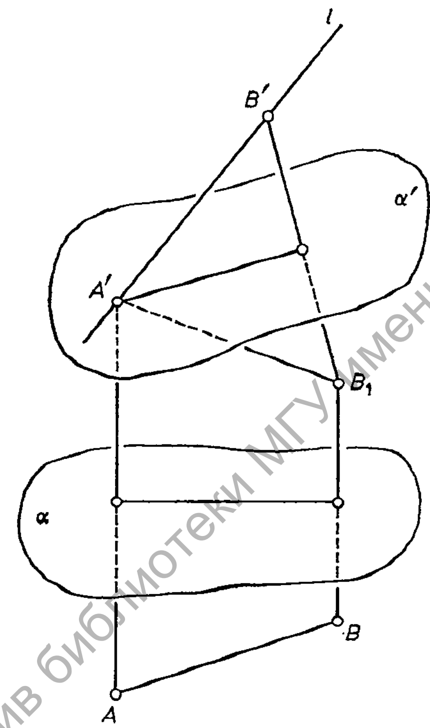

Fig. 3.

Consequence 7. If from the common origin of two rays (which are reflection images of each other with respect to a plane) we lay out two congruent segments, then their endpoints will be reflection images with respect to the given plane.

Consequence 8. Reflection images lie on (a) a perpendicular to the reflection plane, (b) in different half-spaces determined by the reflection plane, (c) at equal distances from the point of intersection of the reflection plane and the perpendicular drawn through the two points.

It is then a very simple matter to prove an important theorem of the theory of reflection at a plane, which ensures the most fundamental applications of this transformation.

* The Russian text has been corrected.

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Theorem 3. The reflection plane of two different points is the locus of points equidistant from the two given points.

Proof. (1) Let $\alpha(A) \equiv A'$ and $P \in \alpha$ (Figure 4). The segments AP and $A'P$ are congruent (Definition 8), and this means that all points on the reflection plane of points A and A' are equidistant from these points.

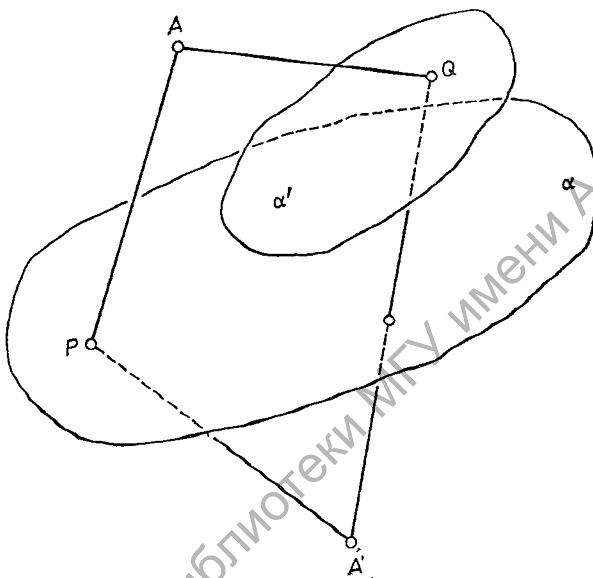

Fig. 4.

(2) Suppose that the point Q does not belong to the plane α. Moreover, let us suppose that the segment AQ is equal to the segment $A'Q$. According to Consequence 7 there is a plane α' such that $\alpha'(A) \equiv A'$, $Q \in \alpha'$. The plane α' cannot coincide with the plane α, since the point Q would then lie in α, which is in conflict with the original choice. However, we then find that the points A and A' are reflection images with respect to two different planes α and α'. This is in conflict with Axiom 4 and, therefore, the segment AQ is not equal to $A'Q$.

By (1) and (2) we have established the validity of the above theorem.

The above material forms the basis for a solid geometry course.

As an illustration of the use of symmetry considerations, let us give a proof of at least one theorem from the grade IX solid geometry course.

Theorem 4. If one of two parallel straight lines is perpendicular to a plane, then the other straight line is also perpendicular to this plane.

Proof. Let $a \perp \alpha$ and $b \parallel a$ (Figure 5). We shall prove that $b \perp \alpha$. Since

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$a \perp \alpha$, we have $\alpha(a) \equiv a$. Suppose that $\alpha(b) \equiv b'$. The parallelism of the two straight lines is invariant under reflection. Therefore $b' \parallel a$. Only one straight line parallel to a can be drawn through the point $A' \equiv b \times \alpha$. Hence $b' \equiv b$, i.e. $\alpha(b) \equiv b$, and this means that $b \perp \alpha$ (Definition 6).

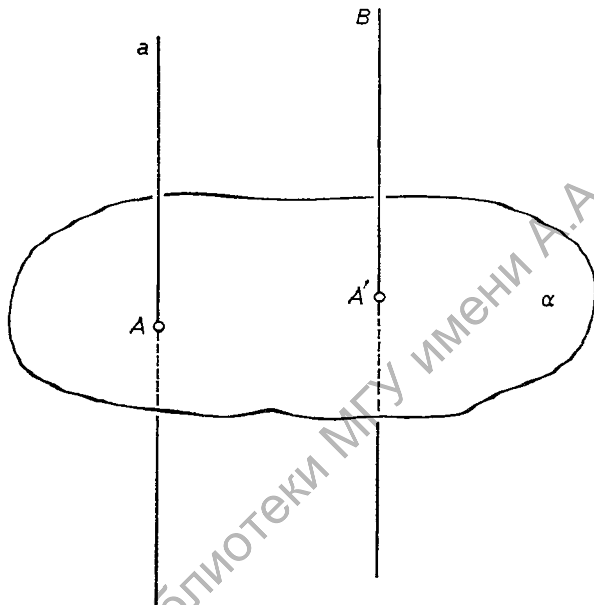

Fig. 5.

It is clear that, in contrast to the "classical" proof in the textbook by A. P. Kiselev, we have not used any additional constructions, and the proof is distinguished by its simplicity and elegance.

The above approach has been used by the author at the V. I. Lenin Boarding School in academic years 1967/68 and 1968/69. The approach was found to be satisfactory, and the use of symmetry considerations in the main course and in the solution of exercises attracted considerable interest and became an important means of mathematical development.

Reflection at a plane was most widely used in connection with the theme of "perpendicularity in space". It was found that the majority of the students assimilated quite well the entire material and acquired correct habits in its application.

The time spent on individual aspects was divided as follows:

(1) Systematization and generalization of ideas from plane geometry on the deductive structure of geometry – 4 hours.

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(2) Joining and order axioms for space – 4 hours.
(3) Axioms on reflection at planes and their consequences – 8 hours.
(4) Parallelism in space – 13 hours.
(5) Perpendicularity in space – 14 hours.

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(Titles translated)

The following books were found to be useful in this project:
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