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INVESTIGATING THE DERIVATIVES OF A COMPOSITE FUNCTION AND THE DERIVATIVES OF THE PRODUCTS OF FUNCTIONS USING THE FUNCTION'S COMPONENTS

We present properties that hold for all the derivatives of composite functions and properties that hold for all the derivatives of products of functions. Using these properties and investigating the derivatives of each component separately, we obtain useful results concerning the high derivatives of both composite functions and products of functions. Based on this material, students can witness the stages of the investigation and perform active investigation by themselves.

Keywords: high-order derivatives, zeroing of derivatives, composite functions, product of functions.

<u>Introduction</u>:

Scientific research, and mathematical research in particular, generally comprises the following stages: (1) Observation of a certain regularity; (2) Posing a hypothesis concerning the existence of a rule that explains the regularity, or which is derived from it; (3) Proving the rule; (4) Using this rule to obtain other results that will help in solving other problems.

In stage 2 it is important to provide a precise mathematical formulation of the rule, while in stages 3 and 4 a complete and precise proof must be provided.

Investigating high-order derivatives provides an opportunity to illustrate these stages.

Let us assume that a composite function or a product function is given, either of which can be differentiated many times. In many cases we would like to answer the following questions: (i) Does the n-th derivative have zeros at a certain point? (ii) What is the sign of the derivative at a certain point?

These questions are relevant in many fields, from the investigation of functions studies in high school (finding maxima and inflection points) and up to advanced subjects in numerical analysis and differential geometry.

Since the number of terms in the derivatives of a composite function and the derivatives of a product of a function increases exponentially, manually checking results for high-order derivatives becomes complicated.

The results we present below allow one to answer the above questions by differentiating each of the components of a function separately, without the need to differentiate the complete composite function or product of functions.

The proof of the results for a composite function is more difficult than the proof for a product of functions, therefore the results for a composite function, including the proofs, can be posited as an example of the investigation process. This would be followed by asking the students to perform a similar process for the product of functions.

Composite function $f(g(x))$

The first derivative of $f(g(x))$ is $f'(g(x))g(x)$; the second derivative is $f''(g(x))g'(x) + f'(g(x))g''(x)$. Simple observation shows that all the derivatives are a sum of addends, where each addend contains a single factor that is a derivative of $f(g(x))$, as well as several factors that are derivatives of $g(x)$. Closer observation shows that there is a relation between the derivative's order and the structure of the components that appear in the derivative. We can formulate this regular pattern in a precise, mathematical way:

Property 1: For the k-th derivative of the function $f(g(x))$, the sum of the differentiations by $g(x)$ in each term is k.

Property 2: For any term in which factor $f(g(x))$ is differentiated n times (i.e., it appears as $f^{(n)}(g(x))$), there will be exactly n factors which are derivatives of $g(x)$. In other words, there will be exactly n factors of the form $g^{(m)}(x)$, $m \in \mathbb{N}$.

These properties can be precisely proven using induction (see [1]).

Using these two properties, we investigate separately the derivatives of components $g(x)$ and $f(g)$ of the composite function and obtain the following results for the derivatives of the composite function $f(g(x))$:

Let the results for the derivative of component $g(x)$ at point x_0 and the derivative of component $f(g)$ at point $g_0 = g(x_0)$ be:	**Then the conclusion concerning the derivatives of $f(g(x))$ at point x_0 is:**
(i) The first $q - 1$ derivatives of $g(x)$ **are equal to zero**, and the q-th derivative **is not equal to zero**; (ii) The first $p - 1$ derivatives of $f(g)$ **are equal to zero**, and the p-**th** derivative **is not equal to zero**.	The first $pq - 1$ derivatives of $f(g(x))$ **are equal to zero** and the pq-th derivative **is not equal to zero**.
(i) The first $q - 1$ derivatives of $g(x)$ **are not equal to zero**; (ii) The derivatives **from q and up to $(p - 1)(q - 1) + 1$** of $g(x)$ **are equal to zero**; (iii) The first $p - 1$ derivatives of $f(g)$ **are not equal to zero**; (iv) The derivatives **from p and up to $(p - 1)(q - 1) + 1$** of $f(g)$ **are equal to zero**; (v) $f(g)$, $g(x)$ and **all** their derivatives are **positive**.	The first $(p - 1)(q - 1)$ derivatives of $f(g(x))$ **are not equal to zero**, and the $(p - 1)(q - 1) + 1$-th derivative **is equal to zero**.
(i) **All** the derivatives of $f(g)$ are **positive**; (ii) **All** the derivatives of $g(x)$ are **positive**	**All** the derivatives of $f(g(x))$ are **positive**.
(i) The **even** derivatives of $f(g)$ are **positive** and the **odd** derivatives of $f(g)$ are **negative**; (ii) **All** the derivatives of $g(x)$ are **negative**	**All** the derivatives of $f(g(x))$ are **positive**.
(i) **All** the derivatives of $f(g)$ are **positive**; (ii) The **even** derivatives of $g(x)$ are **positive** and the **odd** derivatives are **negative**.	All the **even** derivatives of $f(g(x))$ are **positive** and all the **odd** derivatives are **negative**.
(i) **All** the derivatives of $f(g)$ are **negative**; (ii) The **even** derivatives of $g(x)$ are **positive** and the **odd** derivatives are **negative**.	All the **even** derivatives of $f(g(x))$ are **negative** and all the **odd** derivatives are **positive**.

(i) The **even** derivatives of $f(g)$ are **positive** and the **odd** derivatives are **negative**; (ii) The **even** derivatives of $g(x)$ are **negative** and the **odd** derivatives are **positive**.	All the **even** derivatives of $f(g(x))$ are **positive** and all the **odd** derivatives are **negative**.

Example: Which derivative of function $h(x) = (\sin x - 1)^4$ is not equal to zero at point $x = \pi/2$?

If we try to answer this by differentiating and substituting $x = \pi/2$ in each derivative, we shall have to deal with longer and longer expressions at each stage.

$h'(x) = 4(\sin x - 1)^3\cos x$;

$h''(x) = 12(\sin x - 1)^2\cos^2 x - 4(\sin x - 1)^3\sin x$; …;

$h^{(7)}(x) = -2520\cos x\sin^3 x + 5880\cos^3 x\sin x - 5040(\sin x - 1)\sin^2 x\cos x +$
$+ 2184(\sin x - 1)\cos^3 x - 756(\sin x - 1)^2\sin x\cos x - 4(\sin x - 1)^3\cos x$;

$h^{(8)}(x) = 2520\sin^4 x - 30240\sin^2 x\cos^2 x + 8064\cos^4 x -$
$18144(\sin x - 1)\cos^2 x\sin x + 5040(\sin x - 1)\sin^3 x +$
$+ 756(\sin x - 1)^2\sin^2 x - 768(\sin x - 1)^2\cos^2 x + 4(\sin x - 1)^3\sin x$

Substituting $x = \pi/2$ in each derivative, we obtain that the first seven are equal to zero, and the eighth derivative is not.

On the other hand, by using the first rule from the table above, one can easily answer this question by investigating the components $f(g) = g^4$ and $g(x) = \sin x - 1$ separately. For $f(g)$, it is easy to see that at point $g_0 = \sin(\pi/2) - 1 = 0$, the first three derivatives are zero and the fourth derivative is equal to 4!, in other words, $p = 4$.

For $g(x)$, we have $g'(x) = \cos x \Rightarrow g'(\pi/2) = \cos(\pi/2) = 0$,
$g''(x) = -\sin x \Rightarrow g''(\pi/2) = -\sin(\pi/2) = -1 \neq 0$.

We obtained that $g(x)$ is zero for the first derivative and is not zero for the second derivative, i.e., $q = 2$.

According to the rule in the first row, the composite function $h(x) = (\sin x - 1)^4$ becomes zero for the first $pq - 1 = 4\cdot 2 - 1 = 1$ derivatives, and is not zero for the $pq = 4\cdot 2 = 8$-th derivative.

<u>Product of functions $f(x)g(x)$</u>

After the results for a composite function are presented, the students can perform a similar investigation with a product of functions, obtaining the following results: Observing the structure of the components of the derivative of a product function shows that all the derivatives contain components that follow a certain regularity. Let us formulate this regularity precisely:

<u>Property 3</u>: In the k-th derivative of function $f(x)g(x)$, each addend is of the form $f^{(n)}(x)g^{(m)}(x)$, $(n, m \in \mathbb{N})$, and there holds $n + m = k$.

It is important to provide a full and precise proof of this property (by induction). Using this property, if we investigate separately the derivatives of the components $f(x)$ and $g(x)$, we shall obtain the following results for the derivatives of the product function $f(x)g(x)$:

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If the results for the derivative of component $f(x)$ and the derivative of component $g(x)$ at point x_0 are:	**Then the conclusion concerning the derivatives of $f(x)g(x)$ at point x_0 is:**
(i) The first $p - 1$ derivatives of $f(x)$ **are equal to zero**, and the p-th derivative **is not equal to zero**; (ii) The first $q - 1$ derivatives of $g(x)$ **are equal to zero**, and the q-th derivative **is not equal to zero**.	The first $p + q - 1$ derivatives of $f(x)g(x)$ **are equal to zero** and the $p + q$-th derivative **is not equal to zero.**
(i) The first $p - 1$ derivatives of $f(x)$ **are not equal to zero**; (ii) The derivatives **from p and up to $p + q - 1$** of $f(x)$ **are equal to zero**; (iii) The first $q - 1$ derivatives of $g(x)$ **are not equal to zero**; (iv) The derivatives **from q and up to $p + q - 1$** of $g(x)$ **are equal to zero**; (v) $f(x)$, $g(x)$ and **all** their derivatives are of the **same sign**.	The first $p + q - 2$ derivatives of $f(x)g(x)$ **are not equal to zero**, and the $p + q - 1$-th derivative **is equal to zero**.
(i) **All** the derivatives of $f(x)$ are **positive**; (ii) **All** the derivatives of $g(x)$ are **positive**	**All** the derivatives of $f(x)g(x)$ are **positive**.
(i) **All** the derivatives of $f(x)$ are **negative**; (ii) **All** the derivatives of $g(x)$ are **negative**.	**All** the derivatives of $f(x)g(x)$ are **positive**.
(i) **All** the derivatives of $f(x)$ of $g(x)$ have a **fixed sign**; (ii) The derivatives of $f(x)$ of $g(x)$ have **opposite signs**.	**All** the derivatives of f(x)g(x) are **negative**.
(i) **All** the derivatives of $f(x)$ of $g(x)$ have **alternating signs**; (ii) The **even** derivatives of $f(x)$ of $g(x)$ have the **same sign**; (iii) The **odd** derivatives of $f(x)$ of $g(x)$ have the **same sign**.	All the **even** derivatives of $f(x)g(x)$ are **positive** and all the **odd** derivatives are **negative**.

(i) **All** the derivatives of $f(x)$ of $g(x)$ have **alternating signs**; (ii) The **even** derivatives of $f(x)$ of $g(x)$ have **opposite signs**; (ii) The **odd** derivatives of $f(x)$ of $g(x)$ have **opposite signs**.	All the **even** derivatives of $f(x)g(x)$ are **negative** and all the **odd** derivatives are **positive**.

Example:

Is the 11-th derivative of $h(x) = (1/x)\cdot e^{-x}$ positive or negative at point $x_0 = 5$?

This question can be answered using the rule that appears in the sixth row of the last table and by investigating the components $f(x) = 1/x$ and $g(x) = e^{-x}$. separately. $f'(x) = -(1/x^2)$, $f''(x) = 2/x^2$, $f'''(x) = -(6/x)$, $f^{(4)}(x) = 24/x^5 \ldots$

$g'(x) = -e^{-x}$, $g''(x) = t^{-x}$, $f''(x) = -e^{-x}$, $f^{(4)}(x) = e^{-x} \ldots$

It is easy to see that at point $x_0 = 5$ the derivatives of $f(x)$ of $g(x)$ have alternating signs, the even derivatives are positive and the odd derivatives are negative. Therefore, according to the rule in the sixth row, all the even derivatives of $f(x)g(x)$ are positive and all the odd derivatives are negative, and in particular, the 11-th derivative is negative.

References:

1. Moshe Stupel, David Fraivert & Victor Oxman, Investigating derivatives by means of combinatorial analysis of the components of the function, *International Journal of Mathematical Education in Science and Technology*, 2014, 45:6, 892-904, DOI: 10.1080/0020739X.2013.872306